Modular functions & Modular forms

Def let $\mathcal{H} = \{x+iy \in \mathbb{C} : y > 0\}$ be the upper half plane in $\mathbb{C}$.

Def The (full) modular group is the group

$$\Gamma := SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}$$

lemma For $z \in \mathcal{H}$ and $\gamma \in SL_2(\mathbb{Z})$, Γ acts on $\mathcal{H}$ by $\gamma z := \dfrac{az+b}{cz+d}$

proof Group action $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ acts trivially

and $\gamma_2(\gamma_1 z) = (\gamma_2\gamma_1)z$

Also $\gamma z \in \mathcal{H}$

since $\operatorname{Im}(\gamma z) = \operatorname{Im}\left(\dfrac{az+b}{cz+d}\right) = \operatorname{Im}\left(\dfrac{(az+b)(c\bar{z}+d)}{|cz+d|^2}\right)$

$= |cz+d|^{-2} \operatorname{Im}(ad\,z + bc\bar{z})$

$= |cz+d|^{-2} \underbrace{(ad-bc)}_{=1} \operatorname{Im}(z) = \dfrac{\operatorname{Im}(z)}{|cz+d|^2}$ ∎

Def A function $f : \mathcal{H} \to \mathbb{C}$ is called a modular function of weight k

if (1) f is meromorphic on $\mathcal{H}$

(2) For all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $z \in \mathcal{H}$,

$$f(\gamma z) = (cz+d)^k f(z)$$

(3) f is "meromorphic at ∞"

Remark
since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL_2(\mathbb{Z})$ acts trivially on $\mathcal{H}$,
$f(z) = (-1)^k f(z) \Rightarrow \boxed{k \text{ even}}$

Construction Modular fcts Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$

If $\omega_1' = a\omega_1 + b\omega_2$
$\omega_2' = c\omega_1 + d\omega_2$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, then $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$

Interchanging ω_1 & ω_2 if needs we can suppose that $\omega_1/\omega_2 \in \mathcal{H}$,

and the action of $SL_2(\mathbb{Z})$ on lattices corresponds to the action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$

Let $f(\omega_1, \omega_2)$ be a function on the lattices $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ which is homogeneous in Λ in deg $-k \in \mathbb{Z}$

ie $$f(\lambda\omega_1, \lambda\omega_2) = \lambda^{-k} f(\omega_1, \omega_2) \quad \forall \lambda \in \mathbb{C}, \lambda \neq 0.$$

$$\Longrightarrow \quad f(\omega_1, \omega_2) = \omega_2^{-k} f\left(\frac{\omega_1}{\omega_2}, 1\right) = \omega_2^{-k} f(z) \quad (*)$$

where $z = \frac{\omega_1}{\omega_2} \in \mathcal{H}$

This gives $$f(\gamma z) = f\left(\frac{a\left(\frac{\omega_1}{\omega_2}\right) + b}{c\left(\frac{\omega_1}{\omega_2}\right) + d}\right)$$

$$= f\left(\frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2}\right) \overset{(*)}{=} (c\omega_1 + d\omega_2)^k f(a\omega_1 + b\omega_2, c\omega_1 + d\omega_2)$$

$$= (c\omega_1 + d\omega_2)^k f(\omega_1, \omega_2) \quad \text{since same lattice}$$

$$\overset{(*)}{=} \frac{(c\omega_1 + d\omega_2)^k}{\omega_2^k} f\left(\frac{\omega_1}{\omega_2}\right) = (cz+d)^k f(z)$$

ie $f(z)$ satisfies (2) of the definition of a modular form

Recall $$G_k(\omega_1, \omega_2) = \sum_{\omega \in \Lambda}' \frac{1}{\omega^k} = \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n) \neq (0,0)}} \frac{1}{(m\omega_1 + n\omega_2)^k}$$ $k \geq 4$, k even ($G_k = 0$ for k odd)

It is easy to see that it is homogeneous of deg k

and $G_k(\omega_1, \omega_2) = \omega_2^{-k} G_k\left(\frac{\omega_1}{\omega_2}\right)$, where $G_k(z) = \sum_{(m,n) \neq (0,0)} \frac{1}{(mz+n)^k}$

$$\Longrightarrow G_k(\gamma z) = (cz+d)^k G_k(z) \quad (2)$$

(1) the function $G_k(z)$ is holomorphic $\forall z \in \mathcal{H}$ (cvg abs & unif on compacts)

(3) Meromorphic at ∞

Any meromorphic fct on $\mathbb{C}$ which is invariant by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$, ie $\boxed{f(z+1) = f(z)}$, is a well defined function on the strip $\mathbb{C}/T = \{0 \leq z < 1 : z \in \mathbb{C}\}$.

Consider the analytic isomorphism

$$\mathbb{C}/_{\mathbb{Z}} \longrightarrow \mathbb{C}^*$$
$$z \longmapsto e(z) = e^{2\pi i z}$$

We can write $f(z) = g(e(z)) = g(q)$, and $g(q)$ is meromorphic function on $\mathbb{C}^*$.

③ $f(z)$ is meromorphic at $z = \infty$
$\Longleftrightarrow$ $g(q)$ is meromorphic at $q = 0$

Then, $g(q) = \sum_{n=m}^{\infty} a_n q^n$, $a_m \neq 0$ and $\mathrm{ord}_\infty(f) = m$

$$f(z) = \sum_{n=m}^{\infty} a_n e^{2\pi i n z} = \sum_{n=m}^{\infty} a_n e(nz)$$

$\boxed{\text{Fourier coefficients}}$ $\boxed{\text{Fourier expansion at } \infty}$

We want to find the Fourier expansion of $G_k(z)$.

Thm For $k \geq 4$, k even, we have

$$G_k(z) = 2\zeta(k) + 2\frac{(2\pi i)^k}{\Gamma(k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, e(nz)$$ Eisenstein Series

$e^{2\pi i n z} = q^n$

where $\sigma_s(n) = \sum_{d \mid n} d^s$ arithmetically interesting. In particular multiplicative

Then the normalised Eisenstein series $E_k = \dfrac{G_k(z)}{2\zeta(k)}$

has Fourier expansion

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, e(nz)$$

Since for k even $\quad -\dfrac{(2\pi i)^k}{2k!} B_k = \zeta(k)$, where B_k are the Bernoulli numbers

defined as the coefficients in the power series

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}$$

Proof We start from the series

$$(*) \quad \sin(\pi z) = \pi z \prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)\left(1+\frac{z}{n}\right)$$

easy: compare zeroes & poles

Using the logarithmic derivative on both sides of (*), $\frac{d}{dz}\left(\log \sin(\pi z)\right) = \frac{\pi \cos(\pi z)}{\sin(\pi z)}$,

we get

$$\pi \frac{\cos(\pi z)}{\sin(\pi z)} = \frac{d}{dz}\left[\log(\pi z) + \sum_{n=1}^{\infty} \log\left(1-\frac{z}{n}\right) + \log\left(1+\frac{z}{n}\right)\right]$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z-n} + \frac{1}{z+n}$$

related to B_k which are FC of $\frac{x}{1-e^x}$

We have $$\pi\frac{\cos(\pi z)}{\sin(\pi z)} = \pi i\left[\frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}}\right] = \pi i + \frac{2\pi i}{e^{2\pi i z}-1}$$

$$= \pi i - 2\pi i \sum_{d=0}^{\infty} e(dz)$$

Remark for Homework 1: $\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)\left(1+\frac{z}{n}\right)$ explode geometric and we see that the coeffs of z^{k-1} is related to $\zeta(k)$

and differentiating $(k-1)$ times, we have $(k \geq 4)$

$$(k-1)!\sum_{-\infty}^{\infty}(z-n)^{-k} = \frac{d^{k-1}}{dz^{k-1}}\left(-2\pi i\sum_{d=0}^{\infty} e(dz)\right)$$

$(-1)^{k-1}$

$$= \frac{d^{k-2}}{dz^{k-2}}\left(-2\pi i \sum_{d=1}^{\infty} e(dz)(2\pi i d)\right)$$

$$= \dots = -(2\pi i)^k \sum_{d=1}^{\infty} d^{k-1} e(dz)$$

This gives:

$$G_k(z) = \sum_{(m,n)\neq(0,0)} (mz+n)^{k}$$

recall that k is even

$$= \underbrace{2\zeta(k)}_{m=0} + 2\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}(mz+n)^{-k}$$

$-n \to n$, $z \to mz$ and use above

$$= 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{m,d=1}^{\infty} d^{k-1} e(dmz)$$

$n = dm$

$$= 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\underbrace{\left(\sum_{d\mid n} d^{k-1}\right)}_{\sigma_{k-1}(n)} e(nz)$$

Remark $g_2(z) = 60\, G_4(z) =$

$$= \frac{(2\pi)^4}{12}\left[1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\, e(nz)\right]$$ modular of weight 4

$$g_3(z) = 140\, G_6(z)$$

$$= \frac{(2\pi)^6}{216}\left[1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)\, e(nz)\right]$$ modular of weight 6

Then $g_2(z)^3 - 27 g_3(z)^2$ (both modular forms of weight 12)

$$= \underbrace{\left(\frac{16\pi^4}{12}\right)^3 - 27\left(\frac{64\pi^6}{216}\right)^2}_{0} + \sum_{n=1}^{\infty} a(n)\, e(nz)$$

$\Delta(z) = g_2(z)^3 - 27 g_3(z)^2$ is a modular function of weight 12 which vanishes at ∞

Modular fct meromorphic at ∞ $\quad f(z) = \sum_{n \geq N} a_n q^n \quad N \in \mathbb{Z}$

Modular form holomorphic at ∞ $\quad f(z) = \sum_{n \geq 0} a_n q^n$

Cusp form vanishes at ∞ $\quad f(z) = \sum_{n \geq 1} a_n q^n$

$\Delta(z) =$ cusp form of weight 12

$$= (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n)\, e(nz), \quad \tau(1) = 1$$

Ramanujan τ-fct.

The coefficients $\tau(n)$ are arithmetically very interesting!

The modular Group $SL_2(\mathbb{Z})$

Thm The modular group is generated by the 2 matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \& \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$z \mapsto z+1 \qquad\qquad z \mapsto \frac{-1}{z}$$

Then for f a modular fct

$f(Tz) = f(z+1) = f(z)$

$f(Sz) = f\left(\frac{-1}{z}\right) = z^k f(z)$

proof We remark that $\left[\begin{matrix} S^2 = -1 \\ S^4 = 1 \end{matrix}\right]$ & $S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$

$T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $T^n \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+cn & b+dn \\ c & d \end{pmatrix}$ $n \in \mathbb{Z}$

$\Rightarrow ST^n \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a+cn & b+dn \end{pmatrix}$

> Division Algo
> $a = cn + r,$
> $0 \leq r < c$

Claim we can choose $n \in \mathbb{Z}$ (st) $0 \leq a + cn < |c|$, so by induction, we can reduce to $c = 0$, which is always $\pm\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$, and $T^{-m} \pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

by induction, since $T^{-1}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & m-1 \\ 0 & 1 \end{pmatrix}$$

So for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have some combination of S, T, S^{-1}, T^{-1}, say $g(S,T)$, such $g(S,T)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \overset{1 \text{ or } S^2}{\pm}\left[g(S,T)^{-1}\right] \longrightarrow \text{a string of } S, T, S^{-1}, T^{-1}$$

Exercise let $z \in \overline{D}$. Then the stabilizer of z under the action of $SL_2(\mathbb{Z})$ in $\mathfrak{h}$.

$\overline{D} = \{ z = x+iy : |z| \geq 1$ and $-\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2} \}$

$$\mathrm{Stab}(z) = \begin{cases} \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & z \neq e^{2\pi i/3},\ e^{2\pi i/6},\ i \\ \langle S \rangle & z = i \\ \langle ST \rangle \text{ (of order 6)} & z = \omega \\ \langle TS \rangle \text{ (of order 6)} & z = \omega + 1 \end{cases}$$

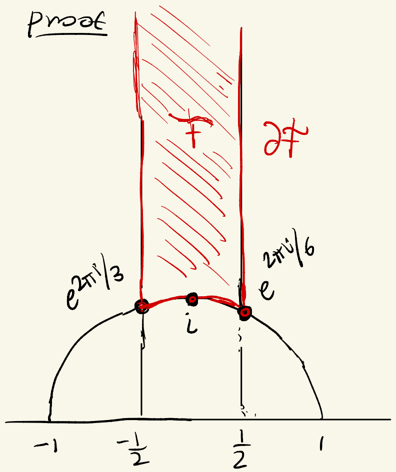
Def A fundamental domain for $SL_2(\mathbb{Z})$ is a set $D \subseteq \mathcal{H}$ such that

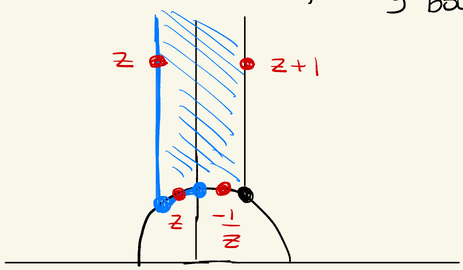
1. D is a domain of $\mathcal{H}$ (non empty, connected & open)
2. Every orbit of $\mathcal{H}/\Gamma$ has a point in D or on ∂D, the boundary of D
3. If $z_1, z_2 \in D$ then they don't belong to the same orbit of Γ, or $z_1 \not\equiv z_2 \mod \Gamma$

Thm Let $D = \{ z = x+iy : |x| < \frac{1}{2}, |z| > 1 \}$.
Then D is a fundamental domain for Γ

Furthermore, the Γ-equivalent points on ∂D are exactly the pairs of points (z, z') where
$z' = z \pm 1$ (on the vertical lines)
or $z' = -\frac{1}{z}$ (on the circular part of boundary).

Proof





Proof Let $z \in \mathcal{H}$. We want to find γz st $\gamma z \in F \cup \partial F$.

$\forall \gamma \quad \mathrm{Im}(\gamma z) = \dfrac{\mathrm{Im}(z)}{|cz+d|^2}$. Since $\forall z$, $\exists$ finitely many $c, d \in \mathbb{Z}$ st $|cz+d| \leq 1$, we can find γ st $|cz+d|$ is minimal ie $|cz+d| \leq |c'z+d'|$ $\forall \gamma' \in \Gamma$

Then $\mathrm{Im}(\gamma z) \geq \mathrm{Im}(\gamma' z)$ $\forall \gamma' \in \Gamma$

Now $\mathrm{Im}(T^n \gamma z) = \mathrm{Im}(\gamma z)$ and choosing n appropriately, we have γ $(= T^n \gamma)$

such that $\mathrm{Re}(\gamma z) \in [-\frac{1}{2}, \frac{1}{2}]$

and $\mathrm{Im}(\gamma z) \geq \mathrm{Im}(S\gamma z)$ $\quad (\gamma' = S\gamma)$

$$= \mathrm{Im}\left(-1/\gamma z\right) = \frac{\mathrm{Im}(\gamma z)}{|\gamma z|^2}$$

$\Rightarrow |\gamma z| \geq 1$.

<u>Furthermore</u> let $z, z' \in \mathcal{F} \cup \partial F$ <u>st</u> $\gamma z = z'$.

wlog $\mathrm{Im}(z') \geq \mathrm{Im}(z) \Leftrightarrow \dfrac{\mathrm{Im}(z)}{|cz+d|^2} \geq \mathrm{Im}(z)$

$\Rightarrow |cz+d|^2 \leq 1$

Since $|cz+d|^2 = |cx+d|^2 + |cy|^2 \leq 1 \quad \circledast$

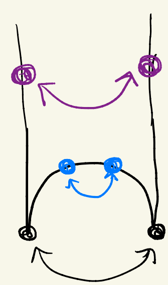
and $y \geq \frac{\sqrt{3}}{2} \Rightarrow |c| \leq 1$

<u>cases</u> $\boxed{c = 0}$ $\begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}$ $z \longmapsto \dfrac{\pm z + b}{\pm 1} = z \pm b$

and both z & $z \pm b \in \mathcal{F} \cup \partial F$

$\Rightarrow b = \pm 1$, and $z' = z \pm 1$ <u>ie</u>

$z = -\frac{1}{2} + iy$ & $z' = \frac{1}{2} + iy$



$\boxed{c = \pm 1} \Rightarrow |d| \leq 1$ by $\circledast$ again

$\boxed{d = 0}$ $\begin{pmatrix} a & \mp 1 \\ \pm 1 & 0 \end{pmatrix}$ & $z' = a - \dfrac{1}{z}$ $\qquad \mathrm{Im}(z') = \dfrac{\mathrm{Im}(z)}{|z|}$

$\mathrm{Re}(z') = a + \mathrm{Re}\left(\frac{-1}{z}\right)$

$= a + \mathrm{Re}\left(\frac{-\bar{z}}{|z|}\right)$

$= a - \mathrm{Re}(z)$

$\Rightarrow |z| = 1$ and $a = 0$ or ± 1

$\boxed{z' = \frac{-1}{z} \text{ and } |z| = 1}$ $\boxed{\text{Fixes } i}$

$\boxed{a = 0}$

$\underline{a = \pm 1}$ (with $c = \pm 1$ and $d = 0$)

$\pm \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad z' = \dfrac{z-1}{z} = 1 - \dfrac{1}{z}$ $\quad \boxed{\text{fixes} \; \omega + 1}$

$\pm \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad z' = \dfrac{-z-1}{z} = -1 - \dfrac{1}{z}$ $\quad \boxed{\text{fixes} \; \omega}$

Similarly, $c = \pm 1$, $d = \pm 1$ leads to

$\pm \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $\pm \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$

$z \to \dfrac{-1}{z+1}$ $\qquad z \mapsto \dfrac{-1}{z-1}$

$\boxed{\text{fixes} \; \omega}$ $\qquad \boxed{\text{fixes} \; \omega + 1}$

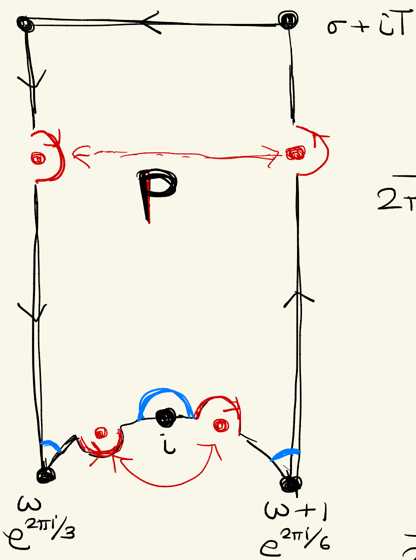
Thm 1.3 Let $f \neq 0$ be a modular form of weight $k \geq 0$.
Let $\omega = e^{2\pi i/3}$. Then $\omega + 1 = e^{2\pi i/6}$

$$\sum_{z \in \mathcal{H}/\Gamma} \mathrm{ord}_z(f) + \mathrm{ord}_z(\infty) + \frac{1}{2}\mathrm{ord}_i(f) + \frac{1}{3}\mathrm{ord}_\omega(f) = \frac{k}{12}$$

or
$$\sum_{z \in \mathcal{H}/\Gamma \cup \{\infty\}} \frac{\mathrm{ord}_z(f)}{W(z)} = \frac{k}{12}$$
where the weights are given by

$$W(z) = \begin{cases} 1 & z \neq i, \omega \\ 2 & z = i \\ 3 & z = \omega \end{cases}$$

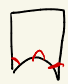
proof



$$\frac{1}{2\pi i}\int_{\partial P} \frac{f'}{f}\, dz = \sum_{z \in P} \mathrm{ord}_z(f)$$

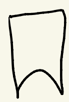
where there are no zeroes on the boundary of P because of our and

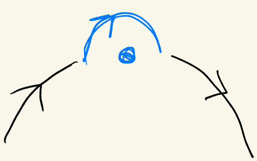
$$\frac{1}{2\pi i}\int_{\partial P} = \int + \int + \int + \int$$



P contains all the singularities on the original boundary inside except $i, \omega, \omega+1$

and we compute the 3 integrals and then $\lim_{\varepsilon \to 0}$, $\lim_{T \to \infty}$:

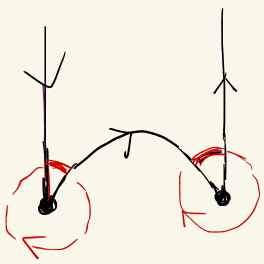


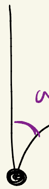


$$\frac{1}{2\pi i}\int \frac{f'}{f}\, dz = -\frac{1}{2}\mathrm{res}_z \frac{f'}{f} = \frac{-\mathrm{ord}_z(f)}{2}$$

(becomes + on the RHS)

If $z = e^{2\pi i/3}$ is a zero of f





what is the angle? $\frac{2\pi}{6} = \frac{\pi}{3}$

$$\Rightarrow \frac{1}{2\pi i}\int \frac{f'}{f}\,dz = \frac{-1}{6}\ \mathrm{ord}_z(\omega)$$

and we will encounter 2 of them.

Then $\frac{1}{2\pi i}\int \frac{f'}{f}\,dz = \frac{-\mathrm{ord}_z(\omega)}{3}$

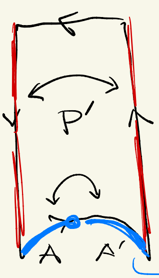
C_ε & C_ε

This gives taking $\varepsilon \to 0$

$$\frac{1}{2\pi i}\int_{P'} \frac{f'}{f}(z)\,dz = \sum_{z \neq i, \omega} \mathrm{ord}_z(f) + \frac{\mathrm{ord}_\omega(f)}{3} + \frac{\mathrm{ord}_i(f)}{2}$$

where P' is the countour:

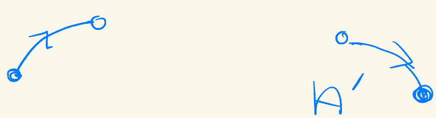
$T \to \infty$



By periodicity $f(z) = f(z+1)$

The red integrals cancel.

$f(\frac{-1}{z}) = z^k f(z)$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$



Differentiating $f(-\frac{1}{z}) = z^k f(z)$, we get

$$f'(-\tfrac{1}{z})(z^{-2}) = kz^{k-1}f(z) + z^k f'(z)$$

and dividing by $z^k f(z)$

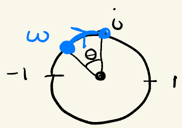
$$\frac{f'(-\frac{1}{z})}{z^k f(z)}\,\frac{1}{z^2} = \frac{k}{z} + \frac{f'(z)}{f(z)}$$

$$\Leftrightarrow \quad \frac{f'}{f}\left(-\frac{1}{z}\right) z^{-2} = \frac{k}{z} + \frac{f'}{f}(z) \quad (*)$$

Then
$$\int_{A'} \frac{f'}{f}(s)\,ds = \frac{-1}{2\pi i}\int_A \frac{f'}{f}\left(\frac{-1}{z}\right)(z^{-2})\,dz \overset{(*)}{=} -\frac{1}{2\pi i}\int_A \frac{k}{z} + \frac{f'}{f}\,dz$$
$$s = -\frac{1}{z}$$

$$\rightarrow \frac{1}{2\pi i}\int_{A\cup A'} \frac{f'}{f}(z)\,dz = \frac{1}{2\pi i}\left(\int_A \frac{f'}{f}\,dz - \int_A \frac{f'}{f}\,dz - \int_A \frac{k}{z}\,dz\right)$$

A



$$\frac{-1}{2\pi i}\int_A \frac{k}{z}\,dz = \frac{k \operatorname{res}_{z=0} \frac{1}{z}}{12} = \frac{k}{12}$$

$$\theta = \frac{\pi}{6} = \frac{2\pi}{12}$$

We then have
$$\frac{1}{2}\operatorname{ord}_i(f) + \frac{1}{3}\operatorname{ord}_w(f) + \sum_{z\neq\{i,w\}} \operatorname{ord}_z(f) = \overset{H}{\circ\!\longleftarrow\!\circ} + \frac{k}{12}$$

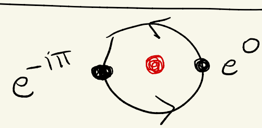
Horizontal segment H: $f(z) = g(e^{2\pi i z})$, $q = e^{2\pi i z}$

$$\frac{f'(z)}{f(z)} = \frac{g'(e^{2\pi i z})\, e^{2\pi i z}\,(2\pi i)}{g(e^{2\pi i z})} \quad \text{and} \quad dq = e^{2\pi i z}(2\pi i)\,dz$$

Replacing in the integral

$$\frac{1}{2\pi i}\int_{z = x+iT} \frac{f'}{f}(z)\,dz = \frac{1}{2\pi i}\int \frac{g'(q)}{g(q)}\,dq = \operatorname{ord}_0(g) = \operatorname{ord}_\infty(f)$$
$$-\frac{1}{2} \leq x \leq \frac{1}{2}$$

$$\underbrace{q = e^{2\pi i x}\, e^{-2\pi y}, \quad -\frac{1}{2} \leq x \leq \frac{1}{2}}_{|q| = e^{-2\pi y}}$$



The linear space of modular forms

$M_k(\Gamma) = \{ f : \mathcal{H} \to \mathbb{C}$ holomorphic on $\mathcal{H}$ and at ∞
and such that $f(\gamma z) = (cz+d)^k f(z) \}$

$E_k(z)$ $k \geq 4$, k even $G_k(z) = \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^k}$, $E_k(z) = \frac{G_k(z)}{2\zeta(k)}$

so that the Fourier expansion at ∞ is $\sum_{n=0}^{\infty} a_n q^n$, $a_0 = 1$

Then $E_k(z) \in M_k(\Gamma)$.

$k=4$ $E_4(\omega) = 0$ (HWK 1)

and by valence formula

$$\frac{4}{12} = \frac{1}{3} + \sum_{\substack{z \in \mathcal{R} \cup \{\infty\} \\ z \neq \omega, i}} \mathrm{ord}_z(E_4) + \frac{\mathrm{ord}_i(E_4)}{2} \implies \text{only one zero}$$

$k=6$ $E_6(i) = 0$ & $E_6(z) \neq 0$ $\forall z \neq i \in \mathcal{R} \cup \{\infty\}$

$\Delta(z) = (60\, G_4(z))^3 - 27\,(140\, G_6(z))^2 \in M_{12}(\Gamma)$ no pole and zero at ∞
$\implies$ no other zero by valence formula.

Ring of all modular forms

$\mathcal{M} = \bigoplus_k M_k(\Gamma)$ and $M_k M_l \subseteq M_{k+l}$ (graded ring).

$\mathcal{M}$ is also a $\mathbb{C}$-vector space (each $M_k(\Gamma)$ is)
So $\mathcal{M}$ is a graded algebra over $\mathbb{C}$.

$$\sum_{\substack{z \neq \{i, \omega\} \\ \text{including } \infty}} \mathrm{ord}_z(f) + \frac{\mathrm{ord}_i(f)}{2} + \frac{\mathrm{ord}_\omega(f)}{3} = \frac{k}{12}$$

Important
$\mathrm{ord}_z(f) \geq 0$
$\forall z$ since f
is holomorphic

① $\underline{k=0}$ $\Rightarrow$ $\underset{\text{only choice}}{\mathrm{ord}_z(f) = 0 \ \forall z}$ $\Rightarrow f \in \mathbb{C}$

② $\underline{k=2}$ impossible

③ $\underline{k=4}$ only choice $\mathrm{ord}_\omega(f) = 1$ & $\mathrm{ord}_z(f) = 0 \ \forall z \neq \omega$

$$M_4 = \mathbb{C}\, G_4(z)$$

④ $\underline{k=6}$ only choice $\mathrm{ord}_i(f) = 1$ & $\mathrm{ord}_z(f) = 0 \ \forall z \neq i$

$$M_6 = \mathbb{C}\, G_6(z)$$

⑤ $\underline{k=8}$ Only choice $\mathrm{ord}_\omega(f) = 2$ & $\mathrm{ord}_z(f) = 0 \ \forall z \neq \omega$

$\frac{8}{12} = \frac{2}{3}$

$$M_8 = \mathbb{C}\, G_4^2(z)$$

⑥ $\underline{k=10}$ Only choice $\mathrm{ord}_\omega(f) = \mathrm{ord}_i(f) = 1$ & $\mathrm{ord}_z(f) = 0$
$\forall z \neq i, \omega$

$\frac{10}{12} = \frac{5}{6}$

$$M_{10} = \mathbb{C}\, G_4(z)\, G_6(z)$$

⑦ $\underline{k=12}$ $\Delta(z) \in M_{12}(\Gamma)$ with a zero at ∞
$\Rightarrow$ only zero by valence formula

For any $f \in M_{12}$

$\boxed{\text{If } f \in M_{12}}$, then $f(z) - c\, G_{12}(z)$ vanishes at ∞ (for the appropriate c)

$$\Rightarrow \frac{f(z) - c\, G_{12}(z)}{\Delta(z)} \in M_0 = \mathbb{C} \quad \text{(no poles)}$$

$$\Rightarrow f(z) = b\, \Delta(z) + c\, G_{12}(z)$$

$$\Rightarrow M_{12} = \mathbb{C}\, \Delta(z) \oplus \mathbb{C}\, G_{12}(z)$$

and by induction $M_k = \Delta(z)\, M_{k-12} \oplus G_k(z)\, \mathbb{C}$ $\quad k \geq 12$

We have proven

Thm $\dim M_k(\Gamma) = \begin{cases} \lfloor k/12 \rfloor & k \equiv 2 \ (12) \\ \lfloor k/12 \rfloor + 1 & k \not\equiv 2 \ (12) \end{cases}$
for k even

Thm Let $S_k(\Gamma)$ be the vector space of cusp forms (holomorphic everywhere and 0 at ∞).
Then the map

$$\begin{aligned} M_k &\longrightarrow S_{k+12} \\ f &\longmapsto f \cdot \Delta \end{aligned}$$

is an isomorphism (of vs),
and $\boxed{\dim S_{k+12} = \dim M_k}$

Furthermore,
$\dim S_k = 0 \quad k \leq 10$
(since $\dim M_k = 1$).

Cor $M = \mathbb{C}[G_4, G_6]$, where G_4 & G_6 are alg. independent

proof First we show that any $f \in M_k$ is a polynomial in G_4 and G_6.
We have proven that for $k \leq 12$. If $k \geq 12$, consider
$g = f - cG_4^a G_6^b$ where $4a + 6b = k$ and c is chosen st g vanishes at ∞. Then $\frac{g}{\Delta} \in M_{k-12}$, and the proof follows by induction.

To show that G_4 & G_6 are alg. ind suppose for a contradiction that $\exists P \neq 0$ st $P(G_4, G_6) = 0$. Then, the monomials in $P(G_4, G_6)$ must have the same weight, and we can write $P(G_4, G_6)$ as a multiple of:

$$G_4^m + G_6 Q(G_4, G_6) = 0$$
$$G_6^m + G_4 Q(G_4, G_6) = 0 \qquad \deg Q < \deg P$$
$$Q(G_4, G_6) = 0$$

Analysing the zeroes, ① & ② are impossible.

Then, by induction, we always get an alg. relation of smaller degree.
Contradiction ∎

Cute identities $E_k(z)E_\ell(z) - E_{k+\ell}(z)$ vanishes at ∞,
so it has to be a cusp form or $= 0$.

$\Rightarrow$ $E_4^2(z) = E_8(z)$ since $\dim S_8 = 0$

$E_4(z)E_6(z) = E_{10}(z)$ since $\dim S_{10} = 0$

$E_6 E_8 = E_4 E_{10} = E_{14}$ since $\dim S_{14} = \dim M_2 = 0$

$E_6^2 - E_{12} = c\,\Delta(z)$ since $\dim S_{12} = 1$

$$\Rightarrow \quad \sigma_7(n) = \sigma_3(n) + 120 \sum_{0<m<n} \sigma_3(m)\sigma_3(n-m)$$

Bernoulli number alert

$$\tau(n) = \frac{65}{756}\sigma_{11}(n) + \frac{691}{756}\sigma_5(n) - \frac{691}{3}\sum_{0<m<n}\sigma_5(m)\sigma_5(n-m)$$

(see HWK 2).